CHAPTER 2

Free Vibration

All systems possessing mass and elasticity are capable of free vibration, or vibration that takes place in the absence of external excitation. Of primary interest for such a system is its natural frequency of vibration. Our objectives here are to learn to write its equation of motion and evaluate its natural frequency, which is mainly a function of the mass and stiffness of the system.

Damping in moderate amounts has little influence on the natural frequency and may be neglected in its calculation. The system can then be considered to be conserva-tive, and the principle of conservation of energy offers another approach to the calcula-tion of the natural frequency. The effect of damping is mainly evident in the diminishing of the vibration amplitude with time. Although there are many models of damping, only those that lead to simple analytic procedures are considered in this chapter.

2.1 VIBRATION MODEL

The basic vibration model of a simple oscillatory system consists of a mass, a massless spring, and a damper. The mass is considered to be lumped and measured in the SI system as kilograms. In the English system, the mass is $m = w/g$ lb · s²/in.

The spring supporting the mass is assumed to be of negligible mass. Its force-deflection relationship is considered to be linear, following Hooke's law, $F = kx$, where the stiffness k is measured in newtons/meter or pounds/inch.

The viscous damping, generally represented by a dashpot, is described by a force proportional to the velocity, or $f = c\dot{x}$. The damping coefficient c is measured in newtons/meter/second or pounds/inch/second.

2.2 EQUATIONS OF MOTION: NATURAL FREQUENCY

Figure 2.2.1 shows a simple undamped spring-mass system, which is assumed to move only along the vertical direction. It has 1 degree of freedom (DOF), because its motion is described by a single coordinate x.

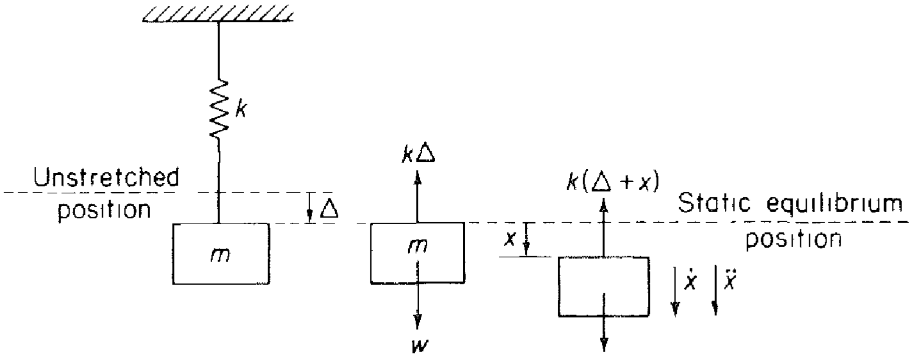

FIGURE 2.2.1. Spring-mass system and free-body diagram.

When placed into motion, oscillation will take place at the natural frequency f_n, which is a property of the system. We now examine some of the basic concepts associated with the free vibration of systems with 1 degree of freedom.

Newton's second law is the first basis for examining the motion of the system. As shown in Fig. 2.2.1 the deformation of the spring in the static equilibrium position is Δ, and the spring force $k\Delta$ is equal to the gravitational force w acting on mass m:

$$k\Delta = w = mg \tag{2.2.1}$$

By measuring the displacement x from the static equilibrium position, the forces acting on m are $k(\Delta + x)$ and w. With x chosen to be positive in the downward direction, all quantities—force, velocity, and acceleration—are also positive in the downward direction.

We now apply Newton's second law of motion to the mass m:

$$m\ddot{x} = \Sigma F = w - k(\Delta + x)$$

and because $k\Delta = w$, we obtain

$$m\ddot{x} = -kx \tag{2.2.2}$$

It is evident that the choice of the static equilibrium position as reference for x has eliminated w, the force due to gravity, and the static spring force $k\Delta$ from the equation of motion, and the resultant force on m is simply the spring force due to the displacement x.

By defining the circular frequency ω_n by the equation

$$\omega_n^2 = \frac{k}{m} \tag{2.2.3}$$

Eq. (2.2.2) can be written as

$$\ddot{x} + \omega_n^2 x = 0 \tag{2.2.4}$$

and we conclude by comparison with Eq. (1.1.6) that the motion is harmonic. Equation (2.2.4), a homogeneous second-order linear differential equation, has the following general solution:

$$x = A \sin \omega_n t + B \cos \omega_n t \tag{2.2.5}$$

where A and B are the two necessary constants. These constants are evaluated from initial conditions $x(0)$ and $\dot{x}(0)$, and Eq. (2.2.5) can be shown to reduce to

$$x = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t \tag{2.2.6}$$

The natural period of the oscillation is established from $\omega_n\tau = 2\pi$, or

$$\tau = 2\pi\sqrt{\frac{m}{k}} \tag{2.2.7}$$

and the natural frequency is

$$f_n = \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{2.2.8}$$

These quantities can be expressed in terms of the statical deflection Δ by observing Eq. (2.2.1), $k\Delta = mg$. Thus, Eq. (2.2.8) can be expressed in terms of the statical deflection Δ as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}} \tag{2.2.9}$$

Note that τ, f_n, and ω_n depend only on the mass and stiffness of the system, which are properties of the system.

Although our discussion was in terms of the spring-mass system of Fig. 2.2.1, the results are applicable to all single-DOF systems, including rotation. The spring can be a beam or torsional member and the mass can be replaced by a mass moment of inertia. A table of values for the stiffness k for various types of springs is presented at the end of the chapter.

EXAMPLE 2.2.1

A $\frac{1}{4}$-kg mass is suspended by a spring having a stiffness of 0.1533 N/mm. Determine its natural frequency in cycles per second. Determine its statical deflection.

Solution The stiffness is

$$k = 153.3 \text{ N/m}$$

By substituting into Eq. (2.2.8), the natural frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{153.3}{0.25}} = 3.941 \text{ Hz}$$

The statical deflection of the spring suspending the $\frac{1}{4}$-kg mass is obtained from the relationship $mg = k\Delta$

$$\Delta = \frac{mg}{k_{\text{N/mm}}} = \frac{0.25 \times 9.81}{0.1533} = 16.0 \text{ mm}$$

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EXAMPLE 2.2.2

Determine the natural frequency of the mass M on the end of a cantilever beam of negligible mass shown in Fig. 2.2.2.

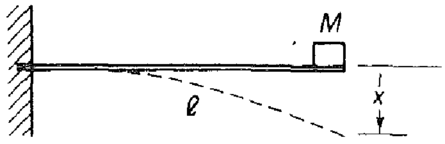

FIGURE 2.2.2.

Solution The deflection of the cantilever beam under a concentrated end force P is

$$x = \frac{Pl^3}{3EI} = \frac{P}{k}$$

where EI is the flexural rigidity. Thus, the stiffness of the beam is $k = 3EI/l^3$, and the natural frequency of the system becomes

$$f_n = \frac{1}{2\pi}\sqrt{\frac{3EI}{Ml^3}}$$

EXAMPLE 2.2.3

An automobile wheel and tire are suspended by a steel rod 0.50 cm in diameter and 2 m long, as shown in Fig. 2.2.3. When the wheel is given an angular displacement and released, it makes 10 oscillations in 30.2 s. Determine the polar moment of inertia of the wheel and tire.

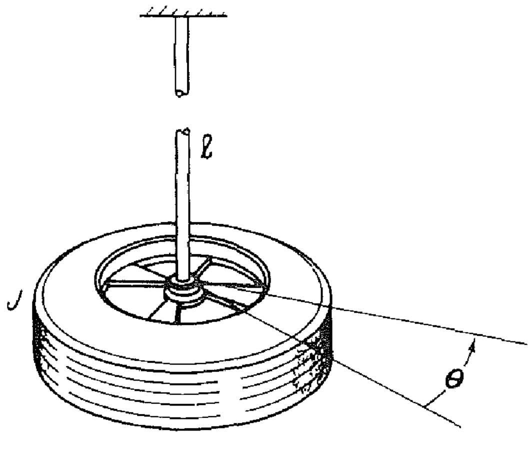

FIGURE 2.2.3.

Solution The rotational equation of motion corresponding to Newton's equation is

$$J\ddot{\theta} = -K\theta$$

where J is the rotational mass moment of inertia, K is the rotational stiffness, and θ is the angle of rotation in radians. Thus, the natural frequency of oscillation is equal to

$$\omega_n = 2\pi\frac{10}{30.2} = 2.081 \text{ rad/s}$$

The torsional stiffness of the rod is given by the equation $K = GI_p/l$, where $I_p = \pi d^4/32 =$ polar moment of inertia of the circular cross-sectional area of the rod, $l =$ length, and $G = 80 \times 10^9 \text{ N/m}^2 =$ shear modulus of steel.

$$I_p = \frac{\pi}{32}(0.5 \times 10^{-2})^4 = 0.006136 \times 10^{-8} \text{ m}^4$$

$$K = \frac{80 \times 10^9 \times 0.006136 \times 10^{-8}}{2} = 2.455 \text{ N}\cdot\text{m/rad}$$

By substituting into the natural frequency equation, the polar moment of inertia of the wheel and tire is

$$J = \frac{K}{\omega_n^2} = \frac{2.455}{(2.081)^2} = 0.567 \text{kg} \cdot \text{m}^2$$

EXAMPLE 2.2.4

Figure 2.2.4 shows a uniform bar pivoted about point O with springs of equal stiffness k at each end. The bar is horizontal in the equilibrium position with spring forces P_1 and P_2. Determine the equation of motion and its natural frequency.

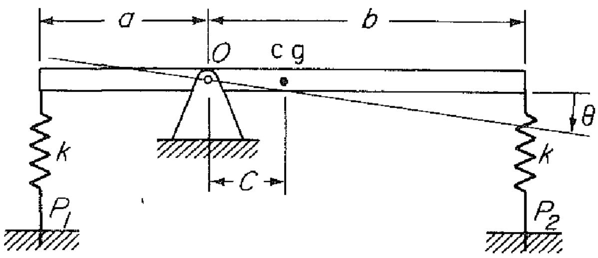

FIGURE 2.2.4.

Solution Under rotation θ, the spring force on the left is decreased and that on the right is increased. With J_O as the moment of inertia of the bar about O, the moment equation about O is

$$\sum M_O = (P_1 - ka\theta)a + mgc - (P_2 + kb\theta)b = J_O\ddot{\theta}$$

However,

$$P_1a + mgc - P_2b = 0$$

in the equilibrium position, and hence we need to consider only the moment of the forces due to displacement θ, which is

$$\sum M_O = (-ka^2 - kb^2)\theta = J_O\ddot{\theta}$$

Thus, the equation of motion can be written as

$$\ddot{\theta} + \frac{k(a^2 + b^2)}{J_O}\theta = 0$$

and, by inspection, the natural frequency of oscillation is

$$\omega_n = \sqrt{\frac{k(a^2 + b^2)}{J_O}}$$

2.3 ENERGY METHOD

In a conservative system, the total energy is constant, and the differential equation of motion can also be established by the principle of conservation of energy. For the free vibration of an undamped system, the energy is partly kinetic and partly potential. The kinetic energy T is stored in the mass by virtue of its velocity, whereas the potential energy U is stored in the form of strain energy in elastic deformation or by a spring or work done in a force field such as gravity. The total energy being constant, its rate of change is zero, as illustrated by the following equations:

$$T + U = \text{constant} \tag{2.3.1}$$

$$\frac{d}{dt}(T + U) = 0 \tag{2.3.2}$$

If our interest is only in the natural frequency of the system, it can be determined by the following considerations. From the principle of conservation of energy, we can write

$$T_1 + U_1 = T_2 + U_2' \tag{2.3.3}$$

where $_1$ and $_2$ represent two instances of time. Let $_1$ be the time when the mass is passing through its static equilibrium position and choose $U_1 = 0$ as reference for the potential energy. Let $_2$ be the time corresponding to the maximum displacement of the mass. At this position, the velocity of the mass is zero, and hence $T_2 = 0$. We then have

$$T_1 + 0 = 0 + U_2 \tag{2.3.4}$$

However, if the system is undergoing harmonic motion, then T_1 and U_2 are maximum values, and hence

$$T_{\max} = U_{\max} \tag{2.3.5}$$

The preceding equation leads directly to the natural frequency.

EXAMPLE 2.3.1

Determine the natural frequency of the system shown in Fig. 2.3.1.

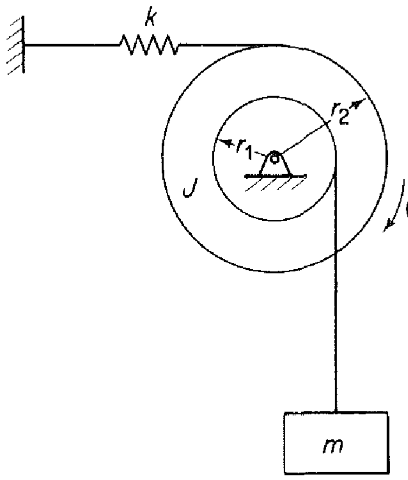

Figure 2.3.1.

Solution Assume that the system is vibrating harmonically with amplitude θ from its static equilibrium position. The maximum kinetic energy is

$$T_{\max} = \left[\tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}m(r_1\dot{\theta})^2\right]_{\max}$$

The maximum potential energy is the energy stored in the spring, which is

$$U_{max} = \tfrac{1}{2}k(r_2\theta)^2_{max}$$

Equating the two, the natural frequency is

$$\omega_n = \sqrt{\frac{kr_2^2}{J + mr_1^2}}$$

The student should verify that the loss of potential energy of m due to position $r_1\theta$ is canceled by the work done by the equilibrium force of the spring in the position $\theta = 0$.

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